TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 232, 1977

ANISOTROPIC H^p, REAL INTERPOLATION, AND FRACTIONAL RIESZ POTENTIALS

BY

W. R. MADYCH

ABSTRACT. We observe that the anisotropic variants of H^p interpolate by the real method in the usual manner. Using this fact we show that the corresponding fractional Riesz potentials and related operators perform an embedding in $H^p, p > 0$, analogous to the one for $L^p, p > 1$. We also state a theorem concerning the mapping properties of $f \to h * f$, where h is in $B_a^{1,\infty}$, which hold only for a restricted range of p.

0. Introduction. Let $\{A_t\}_{t>0}$ be a continuous group of linear transformations on R^n with infinitesimal generator $a, A_t = t^a = \exp(a \log t)$. A tempered distribution, f, on R^n is in $H_a^{p,q}$, $0 < p \leqslant \infty, 0 < q \leqslant \infty$, if and only if $\sup_{t>0}|\phi_t * f(x)|$ is in $L^{p,q}$, where ϕ is in $\mathcal{S}$, $\int\phi \neq 0$, $\phi_t(x) = t^{-\gamma}\phi(A_t^{-1}x)$ and γ is the trace of a.

In the case that a is the identity matrix it is known that

$$(H_a^{p_0,q_0}, H_a^{p_1,q_1})_{\theta,q} = H_a^{p,q}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad 0 < \theta < 1, 0 < q \leqslant \infty,$$

where $(\cdot, \cdot)_{\theta,q}$ denotes the interpolation space by the Lions-Peetre method.

In this note we observe that this result holds for all a satisfying $\langle ax, x\rangle \geqslant \langle x, x\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in R^n. Using this fact we show that the corresponding fractional Riesz potential and related operators perform an embedding in $H_a^p, p > 0$, analogous to the one for $L^p, p > 1$. In §3 we state a theorem concerning the mapping properties of the transformation $f \to k * f$, where k is in $B_a^{1,\infty}$, which hold only for a restricted range of p.

Concerning complex interpolation of the H_a^p spaces, some results have been announced in [1].

I wish to thank Nestor Riviere for suggesting the problem and encouraging me to write it up.

In this paper all functions are complex valued and all integrals are over R^n unless denoted otherwise. As is customary, the symbol C will be used

Received by the editors March 12, 1976.

AMS (MOS) subject classifications (1970). Primary 26A33, 46E99.

© American Mathematical Society 1977

generically to denote constants appearing in certain estimates and need not be the same at different occurrences.

1. **Preliminaries.** Let $\{A_t\}_{t>0}$ be a continuous group of linear transformations on R^n with infinitesimal generator a, $A_t = t^a = \exp(a \log t)$. We assume that a satisfies $\langle ax, x\rangle \geq \langle x, x\rangle$ for all $x \in R^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. If $x \neq 0$, let $\rho(x)$ denote the unique positive number t such that $|A_t^{-1}x| = 1$, $\rho(0) = 0$. It follows that the function $x \to \rho(x)$ is continuous on R^n, is in $C^\infty(R^n \setminus \{0\})$, and satisfies

(i) $\rho(x) = 1$ iff $|x| = 1$.

(ii) $\rho(A_t x) = t\rho(x)$,

(iii) $\rho(x + y) \leq \rho(x) + \rho(y)$,

(iv) $|\{x: \rho(x) \leq r\}| = (\omega_n/n)r^\gamma$,

where γ is the trace of a, ω_n is the area of $\{x: |x| = 1\}$, and if A is a measurable set then $|A|$ denotes its volume. ρ is called the a metric and $B(x, d) = \{y: \rho(x - y) \leq d\}$. For more details concerning this metric see [2], [7], and [12].

Given a function ϕ on R^n, $\pi_t^a\phi$, ϕ_t, and $\pi_t^{a^*}\phi$ are defined by $\pi_t^a\phi(x) = \phi(A_t x)$, $\phi_t(x) = t^{-\gamma}\phi(A_t^{-1}x)$, and $\pi_t^{a^*}\phi(x) = \phi(A_t^* x)$ where a^* and A_t^* are the adjoints of a and A_t, respectively. For a tempered distribution f, $\pi_t^a f$ is defined by $\langle \pi_t^a f, \phi\rangle = \langle f, t^{-\gamma}\pi_{t^{-1}}^a\phi\rangle$ for $\phi \in \mathcal{S}$, where $\langle f, \phi\rangle$ denotes the distribution f acting on ϕ. $\pi_t^{a^*}f$ is defined analogously. f is said to be homogeneous of degree α with respect to a, $-\infty < \alpha < \infty$, if $\pi_t^a f = t^\alpha f$.

$D = (D_1, \ldots, D_n)$ is the usual differential operator $D_i = \partial/\partial x_i$, $i = 1, \ldots, n$. If A is a linear transformation with matrix (a_{ij}) with respect to the usual coordinate system in R^n, then AD is the differential operator with components $(AD)_i = \sum_{j=1}^n a_{ij}D_j$, $i = 1, \ldots, n$. $\bigotimes_m AD$ is the tensor of rank m whose components we index by $\nu_m = (i_1, \ldots, i_m)$, $i_j = 1, \ldots, n$, and $(\bigotimes_m AD)_{\nu_m} = (AD)_{i_1} \cdots (AD)_{i_m}$. Given a tensor, T, of rank m whose components, T_{ν_m}, are numbers, we define the norm of T, $\|T\|$, by the formula $\|T\| = \max_{\nu_m}|T_{\nu_m}|$.

We list several facts which can be easily verified by direct computation.

(1.1) If A and B are linear transformations on R^n then

$$\Big(\bigotimes_m AD\Big)_{\nu_m} f(Bx) = \Big(\bigotimes_m AB^*Df\Big)_{\nu_m}(Bx).$$

(1.2) If f is homogeneous of degree α with respect to a then so is $(\bigotimes_m A^*_{\rho(x)}D)_{\nu_m} f(x)$.

(1.3) $(\pi_t^a f)\hat{} = t^{-\gamma}\pi_{t^{-1}}^{a^*}\hat{f}$ where $\hat{}$ denotes the Fourier transform.

(1.4) If f is homogeneous of degree α with respect to a then $\hat{f}$ is homogeneous of degree $-\alpha - \gamma$ with respect to a^*. Furthermore, if $-\gamma < \alpha < 0$ and if $f \in L^1_{\mathrm{loc}} \cap C^\infty(R^n \setminus \{0\})$, then $\hat{f}$ is in $L^1_{\mathrm{loc}} \cap C^\infty(R^n \setminus \{0\})$.

Given a tempered distribution, f, consider the following:

(1.5) $f^+(x) = \sup_{t>0}|\phi_t * f(x)|$ and $f^+ \in L^p$ for some $\phi \in \mathcal{S}$ with $\int\phi = 1$.

(1.6) $u_N(x) = \sup_{y,t}|\phi_t * f(y)|\,(1 + \rho(x-y)/t)^{-N}$ and $u_N \in L^p$ for some $\phi \in \mathcal{S}$ with $\int\phi = 1$.

(1.7) $f^*(x) = \sup_{\phi\in A_N}\sup_{|x-y|<10t}|\phi_t * f(y)|$ and $f^* \in L^p$ where

$$A_N = \left\{\phi \in \mathcal{S}: \sum_{|\nu|<N}\int (1+\rho(x))^N|D^\nu\phi(x)|\,dx < 1\right\}.$$

THEOREM 1. *If $N > \gamma/p + 1$, $0 < p < \infty$, then* (1.5), (1.6), *and* (1.7) *are equivalent. Furthermore, the L^p norms of f^+, u_N, and f^* are equivalent.*

In the case that a is the identity, this result is contained in [5]. The equivalence of (1.5) and (1.6) is contained in [2, Theorem 2.4]. The proof that (1.6) implies (1.7) is completely analogous to the case when a is the identity (see [5, p. 185]) and we will not repeat it here. The fact that (1.7) implies (1.5) is obvious.

Given a tempered distribution, f, we say that f is in $H_a^{p,q}$ if and only if f^+ is in $L^{p,q}$ and $\|f\|_{H_a^{p,q}} = \|f^+\|_{L^{p,q}}$ where $L^{p,q}$ is the usual Lorentz space. (See [11] for a nice exposition of Lorentz spaces.) Note that $H_a^{p,p} = H_a^p$.

As pointed out in [5] and apparent in [4], f^* is useful for obtaining estimates of $\int f(x)\phi(x)\,dx$ for $\phi \in \mathcal{S}$ and $f \in H_a^p$. More precisely,

$$\left|\int f(x)\phi(x)\,dx\right| \leq N(\phi; x_0, d) \min_{\rho(x_0-y)<10d} f^*(y),$$

where

$$N(\phi; x_0, d) = \min\{A > 0: \phi(x) = A\psi_d(x - x_0),\ \psi \in A_N\}$$

$$\approx \sum_{0<m<N}\sum_{\nu_m}\int\left(1 + \frac{\rho(x-x_0)}{d}\right)^N |(\otimes A_d^* D)_{\nu_m}\phi(x)|\,dx.$$

The following lemmas are basic in the study of these spaces.

LEMMA 1. *Let Ω be an open set properly contained in R^n. There is a collection $\{B(x_i, d_i)\}$ of balls with the following properties*:

(1) *They are pairwise disjoint and $\Omega = \bigcup_{i=1}^\infty B(x_i, 2d_i)$.*

(2) *If Ω^c denotes the complement of Ω, then $B(x_i, 10d_i) \cap \Omega^c \neq \emptyset$ for every i.*

(3) *Let $B_i = B(x_i, 3r_i)$. Then $B_i \subset \Omega$ for all i; and if $x \in \Omega$ and n_x denotes the number of B_i's which contain x then n_x is bounded by a constant, C_0, which depends only on a.*

(4) *Let ψ be a C^∞ function with support in $\{x: \rho(x) < 3\}$ and identically one for $\rho(x) < 2$. If $\psi_i(x) = \psi(A_{d_i}^{-1}(x - x_i))$, then $\chi_\Omega(x) < \Sigma\psi_i(x) < C_0\chi_\Omega(x)$ where C_0 is the constant of* (3). *Furthermore, if $\phi_i(x) = \psi_i(x)[\Sigma_j\psi_j(x)]^{-1}$, then $\Sigma\phi_i = \chi_\Omega$ and $N(\phi_i, x_i, d_i) < C$, where C depends only on a and ψ.*

Lemma 1 is a generalization of the Whitney covering lemma and is a consequence of well-known facts concerning Vitali families in [9, Theorem 2.1]. The construction of the desired covering can be found in the proof of Lemma 1 of [7]. Analogous lemmas have been used by several authors (e.g. see [3]).

LEMMA 2. *Let $f \in H^p \cap L^1$ and $s > 0$ be given. Let $\Omega = \{x: f^*(x) > s\}$ and let $\{B_i\}$ be the covering and $\{\phi_i\}$ the partition of unity of Ω as in Lemma 1. Then f may be written as the sum of two functions, g_s and b_s, where*

(1) $\| g_s\|_{L^\infty} \leqslant Cs$,

(2) $b_s = \sum b_i$ *where*

(i) $b_i(x) = [f(x) - P_i(x)]\phi_i(x)$, *$P_i$ is a polynomial of degree $\leqslant N$ and $\|P_i\|_{L^\infty(B_i)} \leqslant Cs$.*

(ii) $\int P(x)b_i(x)\,dx = 0$ *for all polynomials of degree $\leqslant N$.*

(iii) $\|b_i\|^p_{H^p_a} \leqslant C\int_{B_i}|f^*(x)|^p\,dx$ *and, hence,* $\|b_s\|^p_{H^p_a} \leqslant C\int_\Omega|f^*(x)|^p\,dx$.

This is essentially the main result of [4] in the case that a is the identity. The proof in our case is the same, *mutatis mutandis*.

As in [4], an immediate consequence of Lemma 2 is an interpolation theorem for $H^{p,q}_a$ spaces by the real method. (See [6] for the basic definitions, properties, and consequences.)

THEOREM 2. *For $0 < p_0 < \infty$, $0 < \theta < 1$, and $0 < q \leqslant \infty$, $(H^{p_0}_a, L^\infty)_{\theta,q} = H^{p,q}_a$ where $1/p = (1 - \theta)/p_0$.*

Using reiteration (see [6]) we get the following

COROLLARY. *Suppose θ, p_i, and q_i satisfy $0 < \theta < 1$, and $0 < p_i < \infty$, $0 < q_i \leqslant \infty$, $i = 0,1$. Then $(H^{p_0,q_0}_a, H^{p_1,q_1}_a)_{\theta,q} = H^{p,q}_a$ where $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $0 < q \leqslant \infty$.*

2. **Estimates for Riesz potentials of order** α. The Riesz potential operator of order α, $-\infty < \alpha < \infty$, is the transformation $f \to R_\alpha * f$ where $\hat{R}_\alpha(\xi) = \rho_*(\xi)^{-\alpha}$ and ρ_* is the a^* metric. From (1.4) it follows that if $0 < \alpha < \gamma$ then R_α is in $L^1_{\text{loc}} \cap C^\infty(R^n \setminus \{0\})$ and homogeneous of degree $\alpha - \gamma$ with respect to a. Using (1.2), it follows that $(\bigotimes_m A^*_{\rho(x)}D)R_\alpha(x)$ enjoys the same properties and, hence, $\|(\bigotimes_m A^*_{\rho(x)}D)R_\alpha(x)\| \leqslant C\rho(x)^{\alpha-\gamma}$ for $m = 0, 1, 2, \ldots$, where C is a constant which depends only on m. The above inequality allows us to obtain H^p estimates for $R_\alpha * f$.

LEMMA 3. *Suppose $K \in L^1_{\text{loc}} \cap C^\infty(R^n \setminus \{0\})$ and for some α, $0 < \alpha < \gamma$,*

$$\left\| \bigotimes_m A^*_{\rho(x)} DK(x) \right\| \leqslant C_m \left(\rho(x)\right)^{\alpha-\gamma} \tag{2.1}$$

for all m, where C_m depends only on m. If $\psi \in C^\infty$, $\operatorname{supp}\psi \subset \{x: |x| \leqslant 1\}$, $\int\psi = 1$, and $f \in H^p_a \cap C^\infty$, $0 < p < 1$, then

$$\left|\left\{x: \sup_{\varepsilon>0}|\psi_\varepsilon * K * f(x)| > t\right\}\right| \leqslant \left(Ct^{-1}\|f\|_{H^p_a}\right)^q \tag{2.2}$$

for $t > 0$, where $1/q = 1/p - \alpha/\gamma$ and C is independent of f.

PROOF. First observe that if $2\rho(y) \leqslant \rho(x)$ then

$$\left\|\left(\bigotimes_m A^*_{\rho(x)}D\right)K(x-y)\right\| \leqslant C_m\rho(x)^{\alpha-\gamma} \tag{2.3}$$

for fixed x or for fixed y. Define $K_\varepsilon = \psi_\varepsilon * K$, then K_ε enjoys property (2.1), with C_m independent of ε.

To see this, if $2\varepsilon < \rho(x)$ write

$$\left\|\bigotimes_m A^*_{\rho(x)}DK_\varepsilon(x)\right\| \leqslant \int_{\rho(y)<\varepsilon}|\psi_\varepsilon(y)|\left\|\left(\bigotimes_m A^*_{\rho(x)}D\right)K(x-y)\right\| dy$$
$$\leqslant C_m\|\psi\|_{L^1}\rho(x)^{\alpha-\gamma}.$$

If $2\varepsilon > \rho(x)$, write

$$\left\|\bigotimes_m A^*_{\rho(x)}DK_\varepsilon(x)\right\| \leqslant \int_{\rho(y)<\varepsilon}|K(y)|\left\|\left(\bigotimes_m A^*_{\rho(x)}D\right)\psi_\varepsilon(x-y)\right\| dy.$$

Now,

$$\left\|\left(\bigotimes_m A^*_{\rho(x)}D\right)\psi_\varepsilon(x-y)\right\| = \left\|\left(\bigotimes_m A^*_{\rho(x)/\varepsilon}D\psi\right)_\varepsilon(x-y)\right\|$$
$$\leqslant C\varepsilon^{-\gamma}\left(\frac{\rho(x)}{\varepsilon}\right)^m \max_{1\leqslant j\leqslant m}\sup_x\left\|\bigotimes_j D\psi(x)\right\|$$

where C depends only on m and n. Hence

$$\left\|\bigotimes_m A^*_{\rho(x)}K_\varepsilon(x)\right\| \leqslant C'\varepsilon^{-\gamma}\int_{\rho(y)<\varepsilon}|K(y)|\,dy$$

where C' depends only on m, n, and ϕ. Now $\int_{\rho(y)<\varepsilon}|K(y)|\,dy \leqslant C\varepsilon^\alpha$ and this, together with the previous inequality, implies the desired result.

Suppose that $\|f\|_{H^p_a} = 1$ and let $\Omega = \{x: f^*(x) > s\}$ where $s = t^{q/p}$. Note that $|\Omega| \leqslant Cs^{-p} = Ct^{-q}$. Write $f = g_s + b_s$ as in Lemma 2. Choose p_1 and q_1 so that $1/p_1 - 1/q_1 = \alpha/\gamma$ and $1 < p_1 < \infty$ and $1 < q_1 < \infty$, and recall that the transformation $f(x) \to \sup_{\varepsilon>0}|K_\varepsilon * f(x)|$ maps L^{p_1} into L^{q_1}. Write

$$\left|\left\{x: \sup_{\varepsilon>0}|K_\varepsilon * g_s(x)| > t\right\}\right| \leqslant C\left(t^{-1}\|g_s\|_{L^{p_1}}\right)^{q_1}.$$

Using the fact that $\|g_s\|^{p_1}_{L^{p_1}} \leqslant s^{p_1-p}\|g_s\|^p_{L^p} \leqslant s^{p_1-p}$ and the relations between p, q and p_1, q_1 and s, t, it follows that

$$\left|\left\{x: \sup_{\varepsilon>0}|K * g_s(x)| > t\right\}\right| \leqslant Ct^{-q}. \tag{2.4}$$

Now write

$$I = \left|\left\{x \in \Omega^c\colon \sup_{\varepsilon>0}|K_\varepsilon * b_s(x)| > t\right\}\right| \leqslant t^{-p}\int_{\Omega^c}\left(\sup_{\varepsilon>0}|K_\varepsilon * b_s(x)|\right)^p dx,$$

where Ω^c denotes the complement of Ω. Recall that $b_s = \Sigma b_i$ as in Lemma 2 and the fact that $p < 1$, to write

$$I \leqslant t^{-p}\sum \int_{\Omega^c}\left(\sup_{\varepsilon>0}|K_\varepsilon * b_i(x)|\right)^p dx.$$

To obtain an estimate on $K_\varepsilon * b_i(x)$ assume, without loss of generality, that $x_i = 0$, take x and ε fixed, and write $K_\varepsilon(x - y) = Q(y) + F(x,y)$ where $Q(y)$ is a polynomial of degree less than N and

$$F(x,y) = \int_0^1 \frac{(1-\tau)^{n-1}}{(N-1)!}\ \frac{d^N}{d\tau^N}K_\varepsilon(x - \tau y)\,d\tau.$$

Now, recall that $\int Q(y)b_i(y)\,dy = 0$ and, hence,

$$\begin{aligned}|K_\varepsilon * b_i(x)| &= \left|\int F(x,y)b_i(y)\,dy\right| \\ &\leqslant \left|\int F(x,y)\phi_i(y)f(y)\,dy\right| + \left|\int F(x,y)\phi_i(y)P_i(y)\,dy\right| \\ &= I_1 + I_2.\end{aligned}$$

Since $|P_i(y)| \leqslant Cs$ for $y \in \operatorname{supp}\phi_i \subset \{y\colon \rho(y) \leqslant 3d_i\}$,

$$I_2 \leqslant Cs\int_{\rho(y)<3d_i}|F(x,y)|\,dy = Csd_i^\gamma\int_{\rho(y)<3}|F(x, A_{d_i}y)|\,dy.$$

Observe that

$$\frac{d^N}{d\tau^N}K_\varepsilon(x + \tau y) = \sum_{\nu_N}\left\{\left(\bigotimes_N A^*_{\rho(x)}DK_\varepsilon\right)_{\nu_N}(x - \tau y)\right\}\left(\bigotimes_N A^{-1}_{\rho(x)}y\right)_{\nu_N}$$

and, hence, using 2.3, the fact that

$$\sup_{\rho(y)<3}\left\|\left(\bigotimes_N A^*_{\rho(x)}DK_\varepsilon\right)(x - \tau A_{d_i}y)\right\| \leqslant C\rho(x)^{\alpha-\gamma}$$

and

$$\sup_{\rho(y)<3}\left\|\bigotimes_N A_{d_i/\rho(x)}y\right\| \leqslant C\left[\frac{d_i}{\rho(x)}\right]^N$$

implies that

$$I_2 \leqslant Cs\left[d_i/\rho(x)\right]^{N+\gamma}\rho(x)^\alpha.$$

Recall that

$$I_1 \leqslant N(F(x,\cdot)\phi_i, d_i, 0)\ \min_{\rho(y)<10d_i} f^*(y).$$

To estimate $N(F(x,\cdot)\phi_i, d_i, 0)$, using the change of variables $y \to A_{d_i}y$ in its "formula", it follows that it is bounded by

$$Cd_i^\gamma\left(\max_{0<m<N}\sup_{\rho(y)<3}\left\|\Big(\bigotimes_m D\Big)\Big(\bigotimes_N A^*_{\rho(x)}DK_\varepsilon\Big)(x-\tau A_{d_i}y)\right\|\right)$$
$$\times\left(\max_{0<m<N}\sup_{\rho(y)<3}\left\|\bigotimes_m D\bigotimes_N A_{d_i/\rho(x)}y\right\|\right),$$

where $0 \leqslant \tau \leqslant 1$. Since $\bigotimes_m D = \bigotimes_m A^*_{\rho(x)^{-1}}A_{\rho(x)}D$, we have

$$\left\|\Big(\bigotimes_m D\Big)\Big(\bigotimes_N A^*_{\rho(x)}DK_\varepsilon\Big)(x-\tau A_{d_i}y)\right\|$$
$$=\tau^m\left\|\Big(\bigotimes_m A^*_{d_i/\rho(x)}A_{\rho(x)}D\bigotimes_N A_{\rho(x)}DK_\varepsilon\Big)(x-\tau A_{d_i}y)\right\|$$
$$\leqslant C'\tau^m\left\|A_{d_i/\rho(x)}\right\|^m\left\|\Big(\bigotimes_{N+m} A^*_{\rho(x)}DK_\varepsilon\Big)(x+\tau A_{d_i}y)\right\|$$
$$\leqslant C\rho(x)^{-\gamma+\alpha}$$

for $\rho(y) \leqslant 3$, where C depends only on m and n. Clearly

$$\sup_{\rho(y)<3}\left\|\bigotimes_m D\bigotimes_N A_{d_i/\rho(x)}y\right\| \leqslant C\left(\frac{d_i}{\rho(x)}\right)^N,$$

and since $\min_{\rho(y)\leqslant 10d_i}f^*(y) \leqslant s$, it follows that $I_1 \leqslant Cs(d_i/\rho(x))^{N+\gamma}\rho(x)^\alpha$.

Since the estimates for I_1 and I_2 are independent of ε, we have

$$\sup_{\varepsilon>0}|K_\varepsilon * b_i(x)| \leqslant Csd_i^{N+\gamma}\left(\rho(x-x_i)\right)^{\alpha-N-\gamma}.$$

Hence

$$\int_{\Omega^c}\Big(\sup_{\varepsilon>0}|K_\varepsilon * b_i(x)|\Big)^p\,dx \leqslant \int_{\rho(x-x_i)\geqslant 8d_i}\cdots\,dx \leqslant Cs^p d_i^{\alpha p+\gamma}.$$

Since $|B_i| = Cd_i^\gamma$ and $s^p = t^q$, substituting the last inequality into the estimate for I gives us

$$I \leqslant Ct^{q-p}\sum|B_i|^{1+\alpha p/\gamma} \leqslant Ct^{q-p}\Big(\sum|B_i|\Big)^{1+\alpha p/\gamma}.$$

Since $\Sigma|B_i| \leqslant C'|\Omega| \leqslant Ct^{-q}$, using the relation between p and q it follows that

$$\text{(2.5)}\qquad \left|\Big\{x\in\Omega^c\colon \sup_{\varepsilon>0}|K_\varepsilon * b_s| > t\Big\}\right| \leqslant Ct^{-q}.$$

Inequalities (2.4) and (2.5) imply that

$$\left|\Big\{x\colon \sup_{\varepsilon>0}|K_\varepsilon * f(x)| > 2t\Big\}\right| \leqslant Ct^{-q}$$

where C is a constant independent of f, $\|f\|_{H^p_a} = 1$. Since Lemma 3 follows from the last equality, this completes its proof.

As a consequence of Lemma 3 and Theorem 2 we have

THEOREM 3. *Let f be a distribution in H^p_a, $0 < p < \gamma/\alpha$, and let K and ψ be as in Lemma 3. Then $K * F = \lim_{\varepsilon\to 0} K * \psi_\varepsilon * f$ exists in the sense of distributions, is independent of the choice of ψ, and is in H^q, $1/p - 1/q = \alpha/\gamma$. Furthermore, the mapping $f \to K * f$ is bounded from H^p into H^q, $1/p - 1/q = \alpha/\gamma$.*

In the case a is the identity and $K(x) = |x|^{\alpha-n}$, a variant of Theorem 3 seems to be known. See [8, p. 50].

3. **Remarks.** To avoid introducing more notation, we consider the case where a is the identity matrix. In this case $H^p_a = H^p$.

Recall the defintions of space BMO and the homogeneous Besov spaces $B^{p,q}_\alpha$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $-\infty < \alpha < \infty$. It is well known that the dual of H^p is $B^{\infty,\infty}_{n(1/p-1)}$ if $0 < p < 1$ and BMO if $p = 1$. (A nice account of these facts together with the definitions and properties of the Besov spaces may be found in [10].)

Define $\mathfrak{L}^p$ for $-\infty < 1/p < \infty$ as follows:

$$\begin{aligned} \mathfrak{L}^p &= B^{\infty,\infty}_{-n/p} && \text{if } 1/p < 0,\\ &= \text{BMO} && \text{if } 1/p = 0\\ &= H^p && \text{if } 1/p > 0. \end{aligned}$$

Let $\phi \in \mathcal{S}$ be such that $\hat{\phi} = 0$ in a neighborhood of the origin and a neighborhood of infinity and satisfies $\int_0^\infty \hat{\phi}(t\xi)\, dt/t = 1$ for $\xi \neq 0$. Define $\phi_{\varepsilon,r}$ by

$$\hat{\phi}_{\varepsilon,r}(\xi) = \int_\varepsilon^r \hat{\phi}(t\xi)\, \frac{dt}{t},$$

and note that if $f \in \mathfrak{L}^p$ then $\lim_{\varepsilon\to 0; r\to\infty} \phi_{\varepsilon,r} * f = f$ in $\mathfrak{L}^p$.

THEOREM 4. *Suppose $K \in B^{1,\infty}_\alpha$, $\alpha > 0$, and f is a distribution in $\mathfrak{L}^p$, $1/p \leqslant 1 + \alpha/n$. Then $K * f = \lim_{\varepsilon\to 0; r\to\infty} K * \phi_{\varepsilon,r} * f$ exists in the sense of distributions, is independent of the choice of ϕ, and is in $\mathfrak{L}^q$, $1/p - 1/q = \alpha/n$. Furthermore, the mapping $f \to K * f$ is bounded from $\mathfrak{L}^p$ into $\mathfrak{L}^q$, $1/p - 1/q = \alpha/n$.*

(Note that if K satisfies the hypothesis of Lemma 3 then K is in $B^{1,\infty}_\alpha$. Also observe that it follows from the arguments below that analogous results hold for $K \in B^{p,\infty}_\alpha$, $p > 1$.)

Theorem 4 is well known in the case that $1/p < 1$ (see [10]) and follows by duality for $1 \leqslant 1/p \leqslant 1 + \alpha/n$. In fact, if $(\mathfrak{L}^p, \mathfrak{L}^q)$ denotes the class of translation invariant continuous linear operators mapping $\mathfrak{L}^p$ into $\mathfrak{L}^q$ then, with a slight abuse of notation,

$$B^{1,\infty}_\alpha \subset (H^p, H^1) \subset (\text{BMO}, B^{\infty,\infty}_\alpha) \subset (L^\infty, B^{\infty,\infty}_\alpha) = B^{1,\infty}_\alpha,$$

where $1/p = 1 + \alpha/n$.

The reason the duality argument fails in the case that $1/p > 1 + \alpha/n$ is, of course, the fact that $\mathfrak{L}^q$, $1/q > 1$, is not a Banach space. To see that Theorem 4 cannot be extended to the case $1/p > 1 + \alpha/n$, consider the case $n = 1$, $1/p > 1 + \alpha$, and $1/q = 1/p - \alpha$. Let f and g be C^∞ functions with support in the interval $[-\frac{1}{10}, \frac{1}{10}]$ and define K by the formula $K(x) = \sum_{m=1}^{\infty} m^{-1/q} g(x + m)$. Clearly $K \in B_\alpha^{1,\infty}$ and, if sufficiently many moments of f are zero, f is in H^p. Moreover, if neither f nor g are identically zero,

$$\|K * f\|_{H^q}^q \geq \|K * f\|_{L^q}^q = \|g * f\|_{L^q}^q \sum_{m=1}^{\infty} \frac{1}{m} = \infty.$$

References

1. A. P. Calderon and A. Torchinsky, *Interpolation between H^p spaces, the complex method*, Notices Amer. Math. Soc. **22** (1975), A-166. Abstract #720-42-9.

2. ______, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1-64.

3. R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin and New York, 1971.

4. C. Fefferman, N. M. Riviere and Y. Sagher, *Interpolation between H^p spaces: the real method*, Trans. Amer. Math. Soc. **191** (1974), 75-81.

5. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137-194.

6. T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. **26** (1970), 177-199.

7. W. R. Madych, *On Littlewood-Paley functions*, Studia Math. **50** (1974), 43-63. MR **49** #9536.

8. J. Peetre, *H^p spaces*, Univ. of Lund, Unpublished lecture notes, corrected ed., June 1975.

9. N. M. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243-278.

10. ______, *The Fourier method in approximation theory* (unpublished preprint).

11. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N.J., 1971. MR **46** #4102.

12. A. Torchinsky, *Singular integrals in the spaces $\Lambda(B, X)$*, Studia Math. **47** (1973), 165-190.

Department of Mathematics, Texas A&M University, College Station, Texas 77843

Current address: Department of Mathematics, Iowa State University, Ames, Iowa 50010